Real Analysis

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Theorem 1.

Suppose $\{f_n\}$ *is a sequence of funtions, differentiable on* $[a, b]$ *and such that* $\{f_n(x_0)\}$ *converges for some point* x_0 *on [a,b]. If* $\{f_n'\}$ *converges uniformly on [a,b], then* $\{f_n\}$ *converges uniformly on [a,b], to a function f and* $f'(x) = \lim_{n\to\infty} f_n'(x)$ $(a \leq x \leq b)$.

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Let us now apply Mean Value Theorem to the functions $f_n - f_m$.
Then, $|(f_n(x) - f_m(x)) - (f_n(t) - f_m(t))| \leq |x - t| \frac{\epsilon}{2(b-a)}$ ....(1)

Now

$|f_n(x) - f_m(x)|$
$= |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0)) + f_n(x_0) - f_m(x_0)|$
$\leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)|$
$\leq \frac{\epsilon|x-t|}{2(b-a)} + \frac{\epsilon}{2}$
$< \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$

$\phi_n(t) = \frac{f_n(t)-f_n(x)}{t-x}$

$\phi(t) = \frac{f(t)-f(x)}{t-x}$ for $t \in [a,b]$

$|\phi_n(t) - \phi_m(t)| = \left| f_n(t) - f_n(x) - \frac{f_m(t)-(-f_m(t))}{t-x} \right|$

$\leq \frac{|x-t|}{|t-x|} < \frac{\epsilon}{2(b-a)} \forall m, n \geq N$ and $t \neq x$

$\therefore \{\phi_n(t)\}$ converges uniformly for $t \in [a,b] - \{x\}$

$\lim_{t\to x}\lim_{n\to\infty}\phi_n(t) = \lim_{n\to\infty}\lim_{t\to x}\phi_n(t)$
$\Rightarrow \lim_{t\to x}\lim_{n\to\infty}\frac{f_n(t)-f_n(x)}{t-x} = \lim_{n\to\infty}\lim_{t\to x}\frac{f_n(t)-f_n(x)}{t-x}$
$\lim_{t\to x}\frac{f(t)-f(x)}{t-x} = \lim_{n\to\infty} f_n'(x)$
$f'(x) = \lim_{n\to\infty} f_n'(x).$

Theorem 2 (Theorem7.18).

There exists a real continuous function on the real line which is nowhere diffrentiable.

Define $\phi(x) = |x| \; (-1 \leq x \leq 1)$
Extend ϕ to the whole real line by setting
$\phi(x+2) = \phi(x) \longrightarrow (1)$
Then for all real such that
$|\phi(s) - \phi(t)| = ||s| - |t|| \leq |s - t|$
$\Rightarrow \phi$ is continuous on $\mathbb{R}$

Define $f(x) = \sum_{n=0}^{\infty}(3/4)^n \phi(4^n x) \longrightarrow (2)$
clearly, $|\phi| \leq 1 (0 \leq \phi \leq 1)$
By theorem 7.10
The series converges uniformly on $\mathbb{R}$
Also, the series given in (2) is a series of continuous function.
Hence f is continuous on $\mathbb{R}$.

Claim: Fix a real number x and a positive integer m
put $\delta_m = \pm(1/2)(4^{-m})$
Where the sign is chosen. So that no integer has between
$4^m x$ and $4^m(x + \delta_m)$
Define $\gamma_n = \frac{\phi(4^n(x+\delta_m))-\phi(4^n x)}{\delta_m}$

case(i)

$n > m$

Then $4^n \delta_m$ is an even integer

$4^n \delta_m = \pm(1/2)4^{-m}4^n = \pm(1/2)4^{n-m}$ Therefore $\gamma_n = 0 \forall n > m$

case(ii)

$0 \leq n < m$

$|\gamma_n| = \frac{|\phi(4^n(x+\delta_m)) - \phi(4^n x)|}{|\delta_m|} \leq \frac{|4^n(x+\delta_m) - 4^n(x)|}{|\delta_m|}$ (by(ii)) $=4^n$

case(iii)

$|\gamma_m| = \frac{|(\phi(4^m(x+\delta_m)) - \phi(4^m x)|}{|\delta_m|} \leq \frac{|4^m(x+\delta_m) - 4^m(x)|}{|\delta_m|}$ (by(ii))

$|\gamma_m| = 4^m$ for $n = m$

Now, $\frac{|(f(x+\delta_m) - f(x))|}{|\delta_m|} = \left|\sum_{n=0}^{\propto}(3/4)^n\gamma_n\right|$

$= \left|\sum_{n=0}^{m}(3/4)^n\gamma_n + \sum_{n=m+1}^{\propto}(3/4)^n\gamma_n\right|$

$= \left|\sum_{n=0}^{m}(3/4)^n\gamma_n\right|$?

$= \left|(3/4)^m\gamma_m - (-\sum_{n=0}^{m-1}(3/4)^n\gamma_n\right|$

$\geq 3^m/4^m\,|\gamma_m| - \sum_{n=0}^{m-1}(3/4)^n\,|\gamma_n|$

Therefore $\frac{|f(x+\delta_m) - f(x))|}{|\delta_m|} \geq 3^m - \sum_{n=0}^{m-1}(3^n/4^n)4^n$

$= \frac{3^m - \sum_{n=0}^{m}}{2}$?

$= \frac{3^m+1}{2}$

$\geq \frac{3^m+1}{2}$

$\Rightarrow f'(x)$ does not exist for all $x \in \mathbb{R}$

Therefore f is nowhere diffrentiable (As $m \to\propto \delta_m \to 0$)

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